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A NOTE ON SUBGROUP COMMUTATIVITY DEGREES OF FINITE GROUPS

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Abstract. In this note we give some new results concerning the subgroup commutativity degree of a finite group G. These are obtained by considering the minimum of subgroup commutativity degrees of all sections of G.

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1. Introduction. In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects which have been studied is the probability that two elements of a finite group G commute. It is called the *commutativity degree* of G and is denoted by $d(G)$. Inspired by this concept, in [8] (see also [9]) we introduced a similar notion for the subgroups of G, called the *subgroup commutativity degree* (or the *subgroup permutability degree*) of G. This quantity is defined by

$$\begin{aligned} sd(G) &= \frac{1}{|L(G)|^2} \left|\{(H,K) \in L(G)^2 \mid HK = KH\}\right| = \\ &= \frac{1}{|L(G)|^2} \left|\{(H,K) \in L(G)^2 \mid HK \in L(G)\}\right| \end{aligned}$$

(where $L(G)$ denotes the subgroup lattice of G) and it measures the probability that two subgroups of G commute, or equivalently the probability that the product of two subgroups of G be a subgroup of G.

We recall that for a finite group G we have $sd(G) = 1$ if and only if G is an Iwasawa group, i.e. a nilpotent modular group (see [6, Exercise 3, p. 87]). A complete description of these groups is given by a well-known Iwasawa's result (see Theorem 2.4.13 of [6]). In particular, we infer that $sd(G) = 1$ for all Dedekind groups G.

A well-known result by Gustafson [3] concerning the commutativity degree states that if $d(G) > 5/8$ then G is abelian, and we have $d(G) = 5/8$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the similar problem for the subgroup commutativity degree does not have a solution, i.e. there is no constant $c \in (0,1)$ such that if $sd(G) > c$ then G is Iwasawa, as shows Theorem 2.15 of [1].

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In the following we will study this problem by replacing the condition "$sd(G) > c$" with the stronger condition "$sd^*(G) > c$", where

$$sd^*(G) = \min\{sd(S) \mid S \text{ section of } G\}.$$

It was suggested by the fact that a p-group is modular if and only if each of its sections of order p^3 is. Moreover, if a p-group is not modular then it contains a section isomorphic to D_8, the dihedral group of order 8, or to $E(p^3)$, the non-abelian group of order p^3 and exponent p for $p > 2$ (see Lemma 2.3.3 of [6]). Note that a similar condition for the cyclic subgroup commutativity degree led in [11] to a criterion for a finite group to be an Iwasawa group. We will prove that the condition $sd^*(G) > 23/25$ implies the modularity for finite nilpotent groups G, and also that it implies the solvability for arbitrary finite groups G. Then we will show the non-existence of a constant $c \in (0, 1)$ such that if $sd^*(G) > c$ then G is Iwasawa, extending the above mentioned result of Aivazidis.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7]. For subgroup lattice concepts we refer the reader to [6].

2. Main results.

THEOREM 1. *Let G be a finite nilpotent group such that $sd^*(G) > 23/25$. Then G is modular, and consequently an Iwasawa group.*

Proof. Being nilpotent, G can be written as a direct product of its Sylow subgroups G_i, $i = 1, 2, ..., k$. Clearly, for each i we have

$$sd^*(G_i) \geq sd^*(G) > \frac{23}{25}\,.$$

Assume that G_i is not modular. Then there is a section S of G_i such that $S \cong D_8$ or $S \cong E(p^3)$ for $p > 2$. We can easily check that

$$sd(E(p^3)) = \frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2} < \frac{23}{25} = sd(D_8)$$

and therefore $sd(S) \leq 23/25$, a contradiction. Thus G_i is modular, for all $i = 1, 2, ..., k$, which implies that G is itself a modular group. □

Remark. The constant $23/25$ in Theorem 1 can be decreased for p-groups with $p > 2$ by observing that such a group cannot have sections isomorphic to D_8. Thus, a finite p-group G of odd order which satisfies

$$sd^*(G) > \frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2} \qquad (1)$$

is always an Iwasawa group. We also observe that

$$\frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2} < \frac{3}{p}\,, \text{ for all } p$$

and therefore the above statement remains true by replacing the condition (1) with the more elegant condition

$$sd^*(G) \geq \frac{3}{p}\,.$$

In what follows we will study what can be said about an *arbitrary* finite group G satisfying $sd^*(G) > 23/25$. A first answer is given by the following theorem.

THEOREM 2. *Let G be a finite group such that $sd^*(G) > 23/25$. Then G is solvable.*

Proof. Assume that G is not solvable. Then it contains a section isomorphic to one of the following groups:

- $\mathrm{PSL}(2,p)$, where $p > 3$ is a prime such that $5 \mid p^2+1$;
- $\mathrm{PSL}(2,3^p)$, where $p \geq 3$ is a prime;
- $\mathrm{PSL}(2,2^p)$, where p is a prime;
- $\mathrm{Sz}(2^p)$, where $p \geq 3$ is a prime;
- $\mathrm{PSL}(3,3)$.

It is well-known that $\mathrm{PSL}(2,q)$ has a subgroup isomorphic to D_{q+1} for q odd and a subgroup isomorphic to $D_{2(q+1)}$ for $q = 2^p$ (see e.g. [2]). Then the first three groups above have a section isomorphic to D_8 or to D_{2r} with $r \geq 3$ a prime. But

$$sd(D_{2r}) = \frac{7r+9}{(r+3)^2} < \frac{23}{25}\,,$$

a contradiction. A similar contradiction is obtained for $\mathrm{Sz}(2^p)$ since it contains a subgroup of type $D_{2(2^p-1)}$. Finally, $\mathrm{PSL}(3,3)$ has a subgroup isomorphic to A_4 and

$$sd(A_4) = \frac{16}{25} < \frac{23}{25}\,,$$

contradicting again our hypothesis. This completes the proof. □

Next we try to see whether the condition $sd^*(G) > 23/25$ (or the more general condition $sd^*(G) > c$) implies that G is nilpotent. We start by providing an example of a Schmidt group S_1 of order $p^r q$ for which we are able to compute explicitly $sd^*(S_1)$. It also has the property that if p and q are suitably chosen, then $sd^*(S_1)$ tends to 1 when p tends to infinity. This will be the main ingredient of the proof of Theorem 3.

Example. Let S be a Schmidt group, i.e. a finite non-nilpotent group all of whose proper subgroups are nilpotent. By [5] (see also [4]) it follows that S is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p-subgroup P and a cyclic Sylow q-subgroup Q, and hence S is a semidirect product of P by Q. Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(S)$;
- $Z(S) = \Phi(S) = \Phi(P) \times \langle y^q \rangle$, $S' = P$, $P' = (S')' = \Phi(P)$;
- $|P/P'| = p^r$, where r is the order of p modulo q;
- if P is abelian, then P is an elementary abelian p-group of order p^r and P is a minimal normal subgroup of S;
- if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We infer that $S_1 = S/Z(S)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian p-group P_1 of order p^r by a cyclic group Q_1 of order q (note that S_3 and A_4 are examples of such groups). It is easy to see that S_1 does not contain subgroups of order $p^i q$ for $i = 1, 2, ..., r-1$. Then

$$L(S_1) = L(P_1) \cup \{Q_1^x \mid x \in S_1\} \cup \{S_1\}$$

and so

$$|L(S_1)| = a_{r,p} + p^r + 1\,,$$

where $a_{r,p}$ denotes the total number of subgroups of P_1. By [10] the numbers $a_{r,p}$ can be written as

$$a_{r,p} = f_r(p), \text{ where } f_r \in \mathbb{Z}[\mathrm{X}] \text{ and } \deg(f_r) = \left[r^2/4\right].$$

Since $S_1/1$ is the unique non-abelian section of S_1, one obtains

$$sd^*(S_1) = sd(S_1).$$

Let $Q_1^1, Q_1^2, ..., Q_1^{p^r}$ be the conjugates of Q_1. Then the pairs of commuting subgroups of S_1 are:

- (X, Y) with $X, Y \leq P_1$,
- (X, S_1) and (S_1, X) with $X \leq P_1$,
- $(Q_1^i, 1)$ and $(1, Q_1^i)$, $i = 1, 2, ..., p^r$,
- (Q_1^i, P_1) and (P_1, Q_1^i), $i = 1, 2, ..., p^r$,
- (Q_1^i, S_1) and (S_1, Q_1^i), $i = 1, 2, ..., p^r$,
- (Q_1^i, Q_1^i), $i = 1, 2, ..., p^r$,
- (S_1, S_1).

It follows that

$$\begin{aligned} sd^*(S_1) &= \frac{a_{r,p}^2 + 2a_{r,p} + 7p^r + 1}{(a_{r,p} + p^r + 1)^2} = \\ &= \frac{1 + \dfrac{2}{a_{r,p}} + \dfrac{7p^r}{a_{r,p}^2} + \dfrac{1}{a_{r,p}^2}}{1 + \dfrac{p^{2r}}{a_{r,p}^2} + \dfrac{1}{a_{r,p}^2} + \dfrac{2p^r}{a_{r,p}} + \dfrac{2}{a_{r,p}} + \dfrac{2p^r}{a_{r,p}^2}}\,. \end{aligned} \tag{2}$$

THEOREM 3. *There is no constant $c \in (0,1)$ such that if $sd^*(G) > c$ then G is Iwasawa.*

Proof. Let $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ be two strictly increasing sequences of primes such that the order r_n of p_n modulo q_n is greater than 4. It follows that

$$\left[r_n^2/4\right] > r_n, \forall n \geq 1. \tag{3}$$

For every $n \geq 1$, let G_n be a semidirect product of an elementary abelian p_n-group P_n of order $p_n^{r_n}$ by a cyclic group of order q_n generated by an element x_n which permutes the elements of a basis of P_n cyclically. Then $(G_n)_{n\geq 1}$ are Schmidt groups of order $p_n^{r_n} q_n$ such as S_1 in our example, and so (2) and (3) lead to

$$\lim_{n\to\infty} sd^*(G_n) = 1,$$

completing the proof. $\square$

Inspired by the above results, we came up with the following conjecture.

CONJECTURE 4. *Let G be a finite group such that $sd^*(G) > 23/25$. Then G is either an Iwasawa group or a Schmidt group.*

Remark. The above example also leads to two new classes of finite groups whose subgroup commutativity degree vanishes asymptotically. For $i = 1, 2$, let $(p_n^i)_{n\geq 1}$ and $(q_n^i)_{n\geq 1}$ be two strictly increasing sequences of primes such that the order r_n^i of p_n^i modulo q_n^i is 2 and 3, respectively. Let $(G_n^i)_{n\geq 1}$ be the Schmidt groups of order $(p_n^i)^{r_n^i} q_n^i$ constructed as in the proof of Theorem 3. Then (2) implies that

$$\lim_{n\to\infty} sd^*(G_n^i) = \lim_{n\to\infty} sd(G_n^i) = 0, \; i = 1, 2,$$

as desired.

Finally, we formulate another natural problem concerning our study.

OPEN PROBLEM. *Describe the structure of finite groups G satisfying $sd^*(G) = 23/25$.*

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